AIAA 81-4174

Unsteady Compressible Flow: A Computational Method Consistent with the Physical Phenomena

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A methodology for the numerical prediction of 2-D and 3-D unsteady, inviscid, compressible flows is presented. The primitive Euler equations are recast in terms of compatibility equations on characteristic surfaces. In such a way the evolution in time of the flow properties is described explicitly as the interaction of signals corresponding to the physical wave-propagation phenomenon. The equations are discretized through a finite-difference method where the proper domain of dependence of each computed point is preserved, by approximating the space-derivatives with one-sided differences, according to the velocity of propagation of signals along bicharacteristics. Results of the application of the proposed method to subsonic and transonic flow past airfoils are shown and compared with different methods.

Nomenclature

a = speed of sound
i = incidence
M = Mach number
p = pressure
P = logarithm of pressure
q = flow velocity
R = gas constant
t = time
T = temperature
u,v,w = velocity components
x,y,z = Cartesian coordinates
γ = specific heats ratio
λ = characteristic slope
η,ζ = curvilinear coordinates

All quantities are normalized with respect to reference values:

$$p_{\text{ref}}, T_{\text{ref}}, q_{\text{ref}} = \sqrt{RT_{\text{ref}}}$$

I. Introduction

THIS paper deals with a finite-difference method for solving the equations of the unsteady, inviscid, compressible flow in order to describe time-dependent phenomena per se or to compute steady elliptic or mixed elliptic-hyperbolic flowfields by means of the time-dependent technique. For the sake of simplicity homoentropic flowfields are considered. The significant points of a finite-difference technique are: 1) the choice of the dependent variables and 2) the choice of the numerical scheme. With regard to point 1 there are two representative trends. The first prefers the choice of variables that enable the equations to be written in divergence form.[1] In this way a weak solution can be obtained for shock waves (shock-capturing technique). The second one uses the primitive flow variables (p, q) and is able to recast the equations in form of compatibility relations on characteristic surfaces. In this way the boundaries can be computed correctly, taking into account the waves impinging and reflecting on them.[2] The shock waves can not be captured, and they are explicitly computed as discontinuities according to the Rankine-Hugoniot equations (shock-fitting technique).[3] The main advantage of such a technique is a great accuracy with few computational points, while the codes are fairly complex.

Point 2 was commonly considered less important than point 1 since many good second-order accuracy schemes are available. Some months ago Moretti showed the strong connection between point 1 and point 2. In Ref. 4 he resumes the studies reported in Refs. 5-7 and points out that a finite-difference technique consistent with the physical and mathematical nature of the hyperbolic problems has to integrate the compatibility equations by using a numerical scheme which preserves the domain of dependence of each computed variable. The algorithm proposed by Moretti (λ scheme), denotes a shock-capturing capability without the wiggles usually connected with the conservative shock-capturing techniques.

This paper aims to go deep into the analysis reported in Refs. 4-7 and proposes a method based on the bicharacteristics of multidimensional flowfields, which uses variables that are defined along bicharacteristics, and space-derivatives that are components of derivatives along bicharacteristics. More details can be found in Ref. 8.

II. Theory

We next treat separately the cases for one-, two-, and three-dimensional flow.

One-Dimensional

For the sake of simplicity, let us consider the one-dimensional case. The equations of motion are

$$P_t + uP_x + \gamma u_x = 0 \qquad u_t + uu_x + TP_x = 0 \tag{1}$$

They are hyperbolic and can be recast into compatibility equations along the characteristic directions $\lambda^+ = u + a$, $\lambda^- = u - a$

$$\begin{aligned} (u_t + \lambda^+ u_x) + (a/\gamma)(P_t + \lambda^+ P_x) = 0 \\ (u_t + \lambda^- u_x) - (a/\gamma)(P_t + \lambda^- P_x) = 0 \end{aligned} \tag{2a}$$

With reference to Fig. 1, let us assume that the flowfield is given at time t, while the point N has to be computed at time $t + \Delta t$. From a general point of view a finite-difference method consists in approximating by finite difference the space derivatives u_x, P_x and then computing the time derivatives u_t, P_t by Eqs. (1). The integration in time of u_t and P_t provides the new values u, P at time $t + \Delta t$.

The time derivatives u_t, P_t are physically due to the interaction of the waves that impinge on point N at time t. The wavepaths are the characteristic λ^+; λ^- (Fig. 1). An algorithm consistent with the physical phenomenon must take into account that each wavefront experiences only the flowfield behind itself, and is unaffected by the points in front of it.

Received July 28, 1980; revision received Feb. 9, 1981.

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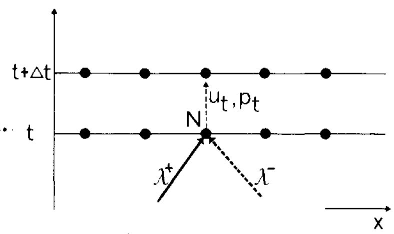

Fig. 1 One-dimensional computational plane sketch.

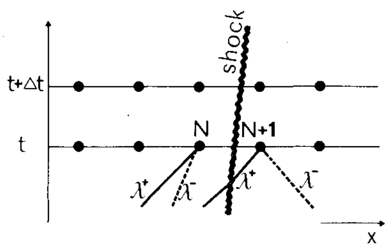

Fig. 2 One-dimensional computational plane sketch.

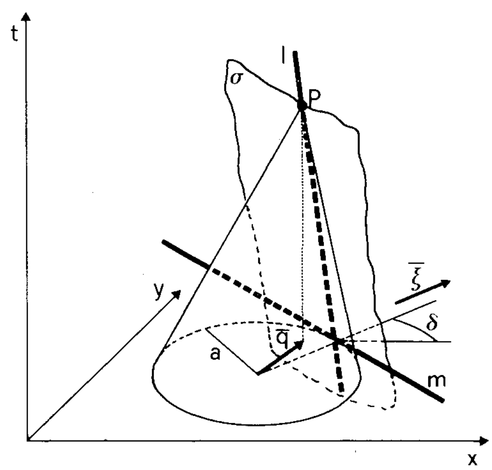

Fig. 3 Characteristic cone.

According to the preceding analysis, let us rewrite the compatibility Eqs. (2a)

$$\begin{aligned}(u_t+\lambda^+u_x^+)+(a/\gamma)(P_t+\lambda^+P_x^+)&=0\\(u_t+\lambda^-u_x^-)-(a/\gamma)(P_t+\lambda^-P_x^-)&=0\end{aligned}\tag{2b}$$

where two kinds of space derivatives appear (u_x^+, P_x^+) and (u_x^-, P_x^-). These are evaluated by backward or forward differences, depending on the sign of the coefficients $\lambda^{\pm}$, that is the characteristic propagation direction. This procedure is physically consistent, because the space derivatives may be discontinuous across a wavefront. Furthermore, it outcomes as a logical procedure if we take into account the Riemann invariants of the flow

$$C=u+\frac{2}{\gamma-1}a \qquad D=u-\frac{2}{\gamma-1}a \tag{3}$$

The compatibility equations can be written now

$$C_t+\lambda^+C_x=0 \qquad D_t+\lambda^-D_x=0 \tag{4}$$

These equations clearly show how the two waves carry distinctly their own contents (C, D). It is also remarked that each variable requires derivatives along one characteristic.

Equations (4) account for the λ scheme shock-capturing capability. With reference to Fig. 2 let us assume that the flow is supersonic at point N and turns subsonic at point $N+1$ by crossing a shock wave. According to the λ scheme, the space derivatives at N are evaluated by backward differences and the point N is unaffected by the shock. With regard to point $N+1$, the derivatives C_x and D_x are defined by backward (that is across the shock) and forward differences, respectively. The actual variable C behaves like the entropy and experiences a discontinuity which depends on the shock strength, negligible for a weak shock.[9] On the contrary, the variable D is strongly discontinuous across the shock. The numerical scheme looks upon C as a continuous variable, while avoiding any difference of D across the shock and saving its discontinuity.

A shock capture[4] is reached also by using the λ scheme associated with the primitive flow variables u, P, although they are strongly discontinuous across the shock. This fact can be explained by considering that only the space derivatives u_x^+, P_x^+ are evaluated across the shock and then combined according to Eq. (2b) so that actually the derivative of the Riemann invariant C is computed

$$u_x^+ + (a/\gamma)P_x^+ = C_x \tag{5}$$

Let us point out that relation (5) is not a numerical identity because of the nonlinear term a, which nevertheless undergoes small variations. However, in our opinion, the direct use of the Riemann invariants is to be preferred because of its consistency with the λ-scheme shock-capturing behavior and because, from a practical view point, it allows for simple codes and fewer space derivatives.

Two-Dimensional Case

The extension of the one-dimensional analysis to the two-dimensional case is not direct. Again the numerical scheme should be consistent with the hyperbolic feature of the problem; it should take into account the signals that separately reach a computational point from its surroundings.

It has been shown formerly that in the one-dimensional flowfield two signals are carried along the characteristic directions and they define the two needed time derivatives (u_t and P_t, or, by using the Riemann invariants, C_t and D_t). In the two-dimensional case, a point is reached by an infinity of signals whose paths are the "rays" or "bicharacteristics" converging on it. Therefore, an infinity of compatibility equations is available to compute the three needed dependent variables (for instance u, v, P). Moreover, the compatibility equations involve derivatives along two directions that have to be chosen on a characteristic surface. The proposed method is based, necessarily, on a discrete number of bicharacteristics. According to Butler[10] we use four compatibility equations involving derivatives along the bicharacteristic direction and along the intersection of the characteristic surface with a space-like surface. These equations can be combined together in order to retain only derivatives along the bicharacteristic paths.

Figure 3 shows the characteristic conoid that reaches point P. The line m is the intersection of the characteristic surface σ touching the conoid on the bicharacteristic l, with a space-like surface (for instance, a $t=$const plane). The equations of motion are as follows.

Continuity:

$$P_t+\bar{q}\cdot\nabla P+\gamma\nabla\cdot\bar{q}=0 \tag{6a}$$

Momentum:

$$\bar{q}_t+(\bar{q}\cdot\Delta)\bar{q}+T\Delta P=0 \tag{6b}$$

The compatibility equations on characteristic surfaces can be obtained by combining Eqs. (6).

$$[\bar{q}_t + (\bar{q}\cdot\nabla)\bar{q} + T\nabla P]\cdot\xi + (a/\gamma)[P_t + \bar{q}\cdot\nabla P + \gamma\nabla\cdot\bar{q}] = 0 \tag{7}$$

where ξ is the unit vector with direction normal to m. Equation (7) is recast in form of compatibility equation

$$\cos\delta\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)_\ell + \sin\delta\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_\ell + \frac{a}{\gamma}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_\ell + a\left(\cos\delta\frac{\partial v}{\partial m} - \sin\delta\frac{\partial u}{\partial m}\right) = 0 \tag{8}$$

where $(\mathrm{d}/\mathrm{d}t)_\ell$ denotes an intrinsic derivative along a bicharacteristic; that is, in a Cartesian reference frame

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_\ell = \frac{\partial}{\partial t} + (u + a\cos\delta)\frac{\partial}{\partial x} + (v + a\sin\delta)\frac{\partial}{\partial y} \tag{9}$$

and $\partial/\partial m$ is

$$\frac{\partial}{\partial m} = -\sin\delta\frac{\partial}{\partial x} + \cos\delta\frac{\partial}{\partial y}$$

The parameter δ is the angle between ξ and the x axis, it denotes the single infinity of characteristic surfaces through point P. By chosing $\delta_1 = 0$, $\delta_2 = \pi$, $\delta_3 = \pi/2$, $\delta_4 = (3/2)\pi$, four compatibility equations are obtained.

$$\begin{aligned}
&\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)_{\ell_1} + \frac{a}{\gamma}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_{\ell_1} = -av_y\\
&\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)_{\ell_2} - \frac{a}{\gamma}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_{\ell_2} = av_y\\
&\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_{\ell_3} + \frac{a}{\gamma}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_{\ell_3} = -au_x\\
&\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_{\ell_4} - \frac{a}{\gamma}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_{\ell_4} = au_x
\end{aligned} \tag{10}$$

Furthermore, we define four variables that are analogous with the Riemann invariants for the one-dimensional flow.

$$C = u + \frac{2}{\gamma - 1}a, \qquad D = u - \frac{2}{\gamma - 1}a,$$
$$E = v + \frac{2}{\gamma - 1}a, \qquad F = v - \frac{2}{\gamma - 1}a \tag{11}$$

Equations (10) can be written

$$\left(\frac{\mathrm{d}C}{\mathrm{d}t}\right)_{\ell_1} = -av_y \qquad \left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\ell_3} = -au_x$$
$$\left(\frac{\mathrm{d}D}{\mathrm{d}t}\right)_{\ell_2} = av_y \qquad \left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)_{\ell_4} = au_x \tag{12}$$

where each of the variables (11) is related to one bicharacteristic. We retain only the space-derivatives of such variables and neglect all derivatives along m. Then, we add the first to the second and the third to the fourth of Eqs. (12). We obtain two equations which provide the time derivatives u_t, v_t

$$\begin{aligned}
u_t &= -\tfrac{1}{2}[(u+a)C_x + (u-a)D_x + vC_y + vD_y]\\
v_t &= -\tfrac{1}{2}[uE_x + uF_x + (v+a)E_y + (v-a)F_y]
\end{aligned} \tag{13}$$

The third equation needed is obtained in a way similar to Butler.[10] The four Eqs. (12) are combined together, providing

$$4P_t - (\gamma/a)[u(E_x - F_x) + v(C_y - D_y) + (u+a)C_x - (u-a)D_x + (v+a)E_y - (v-a)F_y] = -2\gamma(u_x + v_y) \tag{14}$$

The continuity Eq. (6) can be written

$$P_t + \tfrac{1}{2}[u(E_x - F_x) + v(C_y - D_y)] + \gamma(u_x + v_y) = 0 \tag{15}$$

By combining Eqs. (14) and (15), the time derivative of the pressure is expressed by

$$P_t = -\tfrac{1}{2}(\gamma/a)[(u+a)C_x - (u-a)D_x + (v+a)E_y - (v-a)E_y] \tag{16}$$

The computation is carried on by means of Eqs. (13) and (16) where the space derivatives are evaluated by one-sided differences according to the sign of the components of the velocity of propagation along the bicharacteristics ($u+a$, u, $u-a$, $v+a$, v, $v-a$). In the most general case, a curvilinear reference frame is used

$$\zeta = \zeta(x,y); \qquad \eta = \eta(x,y) \tag{17}$$

The space-derivatives in Eqs. (13) and (16) are recast by using

$$\frac{\partial}{\partial x} = \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} \qquad \frac{\partial}{\partial y} = \frac{\partial \zeta}{\partial y}\frac{\partial}{\partial z} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} \tag{18}$$

The one-sided derivative with respect to the curvilinear coordinates depend on the sign of the contravariant components of the velocity of propagation along the bicharacteristic to which each variable is related:

$$\begin{aligned}
&(A + a\zeta_x) \text{ for } C_\zeta && (A + a\zeta_y) \text{ for } E_\zeta\\
&(B + a\eta_x) \text{ for } C_\eta && (B + a\eta_y) \text{ for } E_\eta\\
&(A - a\zeta_x) \text{ for } D_\zeta && (A - a\zeta_y) \text{ for } F_\zeta\\
&(B - a\eta_x) \text{ for } D_\eta && (B - a\eta_y) \text{ for } F_\eta
\end{aligned} \tag{19}$$

where A and B are the contravariant components of the flow velocity

$$A = u\zeta_x + v\zeta_y \qquad B = u\eta_x + v\eta_y \tag{20}$$

For the computation on a time-like surface where boundary conditions are prescribed, we use a weighted summation of three compatibility Eqs. (12), as suggested by Butler.[10] Let us suppose, for instance, that the surface $y = \text{const}$ is a solid wall in a Cartesian reference frame. The boundary condition prescribes the normal velocity component to be zero ($v = 0$). The velocity component u is computed by means of the first of Eqs. (13). The equation for the pressure is obtained by a combination of the first and the second of Eqs. (12), together with twice of either of the last two (the one corresponding to the bicharacteristic that reaches the wall from the flowfield). For instance

$$2\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\ell_3} + \left(\frac{\mathrm{d}C}{\mathrm{d}t}\right)_{\ell_1} - \left(\frac{\mathrm{d}D}{\mathrm{d}t}\right)_{\ell_2} = 2a(u_x + v_y) \tag{21}$$

Then, by means of Eq. (15), one has

$$P_t = -\tfrac{1}{2}(\gamma/a)[2aE_y + (u+a)C_x - (u-a)D_x] \tag{22}$$

In a general curvilinear reference frame, the computation on a solid wall may be carried on a similar way, by defining the local variables

$$\tilde{C}=\tilde{u}+\frac{2}{\gamma-1}a; \qquad \tilde{D}=\tilde{u}-\frac{2}{\gamma-1}a;$$
$$\tilde{E}=\tilde{v}+\frac{2}{\gamma-1}; \qquad \tilde{F}=\tilde{v}-\frac{2}{\gamma-1}a \tag{23}$$

where $\tilde{u}$ and $\tilde{v}$ are the velocity components normal and tangential to the wall.

Three-Dimensional Case

The analysis carried out for the two-dimensional case can be extended directly to the three-dimensional time-dependent flow. Now we are dealing with a problem in four independent variables (x, y, z, t); the characteristic surfaces are Riemannian three-dimensional varieties, then the compatibility equations involve derivatives along three directions. Let such directions be defined by the bicharacteristic and by two directions on the intersection of the characteristic variety with the space-like variety $t=$ cost. By extending the procedure adopted for the two-dimensional case, six compatibility equations are used. If a Cartesian frame of reference is used, the equations are

$$\left(\frac{dC}{dt}\right)_{\ell_1}=-a(v_y+w_z) \qquad \left(\frac{dD}{dt}\right)_{\ell_2}=a(v_y+w_z)$$
$$\left(\frac{dE}{dt}\right)_{\ell_3}=-a(u_x+w_z) \qquad \left(\frac{dF}{dt}\right)_{\ell_4}=a(u_x+w_z)$$
$$\left(\frac{dG}{dt}\right)_{\ell_5}=-a(u_x+v_y) \qquad \left(\frac{dH}{dt}\right)_{\ell_6}=a(u_x+v_y) \tag{24}$$

where

$$G=w+[2/(\gamma-1)]a; \qquad H=w-[2/(\gamma-1)]a \tag{25}$$

and

$$\left(\frac{d}{dt}\right)_{\ell_{1/2}}=(u\pm a)\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}+\frac{\partial}{\partial t}$$
$$\left(\frac{d}{dt}\right)_{\ell_{3/4}}=u\frac{\partial}{\partial x}+(v\pm a)\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}+\frac{\partial}{\partial t}$$
$$\left(\frac{d}{dt}\right)_{\ell_{5/6}}=u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+(w\pm a)\frac{\partial}{\partial z}+\frac{\partial}{\partial t} \tag{26}$$

Just as for the two-dimensional case, the four equations needed are obtained by combining the compatibility equations [Eqs. (24)]

$$u_t=-½[(u+a)C_x+(u-a)D_x+v(C_y+D_y)+w(C_z+D_z)]$$
$$v_t=-½[u(E_x+F_x)+(v+a)E_y+(v-a)F_y+w(E_z+F_z)]$$
$$w_t=-½[u(G_x+H_x)+v(G_y+H_y)+(w+a)G_z+(w-a)H_z]$$
$$P_t=-½\frac{\gamma}{a}[(u+a)C_x-(u-a)D_x+(v+a)E_y-(v-a)F_y+(w+a)G_z-(w-a)H_z] \tag{27}$$

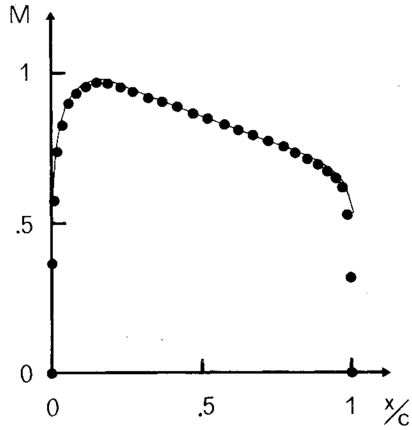

Fig. 4 Mach number distribution on NACA 0012 airfoil: $i=0$ deg, $M_\infty=0.72$; circles, present method; solid line, Ref. 11.

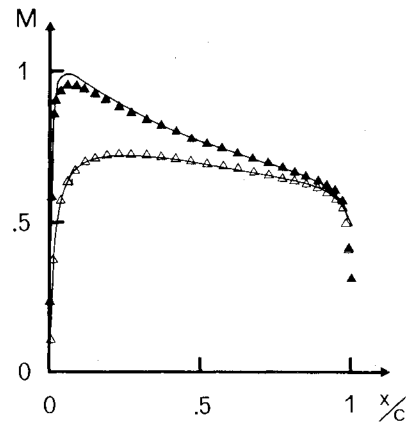

Fig. 5 Mach number distribution on NACA 0012 airfoil: $i=2$ deg, $M_\infty=0.63$; triangles, present method; solid lines, Ref. 11.

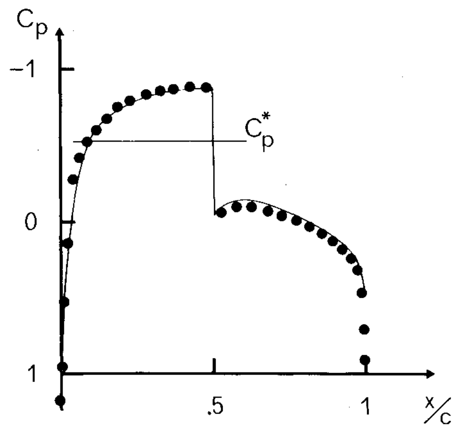

Fig. 6 Pressure coefficient distribution on NACA 0012 airfoil: $i=0$ deg, $M_\infty=0.8$; circles, present method; solid lines, conservative solution of full potential flow equation.

III. Numerical Examples

The proposed method has been tested by computing the flowfield past the NACA 0012 airfoil in a steady subcritical case as well as in a supercritical case according to the time-dependent technique. The airfoil has been conformally mapped inside the unit circle. We use 60 mesh points around the body and 15 mesh points between the airfoil and the point at infinity, a network coarser than the ones usually required by other methods. Figures 4 and 5 show the computed Mach numbers on the airfoil for two subcritical cases: $M_\infty=0.72$, $i=0$ deg, and $M_\infty=0.63$, $i=2$ deg.

These results are compared with those reported by Lock in Ref. 11. Figure 6 shows the pressure coefficient C_p computed in the supercritical case, $M_\infty = 0.8$, $i = 0$ deg. A comparison is done with the results obtained by using the fully conservative relaxation technique for the potential flow equation. The comparisons denote fairly good agreements. In the case of Fig. 6, the proposed method shows its shock-capturing capability; the shock is captured in one mesh without wiggles and its location well matches the one computed by means of a completely different technique.

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